MIXED OPTIMIZATION TECHNIQUE FOR LARGE-SCALE WATER-RESOURCE SYSTEMS

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ABSTRACT: Dealing with significantly large design problems for water-resource systems a mixed optimization procedure based on network linear programming and the subgradient method will be described. Using a linear problem formulation, the procedure uses network linear programming as a subproblem that assumes the knowledge of design variables. Since inside its domain, the global objective function is a convex piecewise linear function, a subgradient method is used to obtain the direction of the improvement of design variables at each iteration using the solutions of the network subproblem. The mixed technique permits an efficient evaluation of the design variables in order to reach a good approximation of the global objective function optimum. The solution technique performs well in the purely linear case and, moreover, allows some kinds of nonlinearities in the cost functions of design variables.

INTRODUCTION

Unlike most other optimization techniques, network linear programming permits one to solve minimal cost network flow problems even when dealing with significantly large multireservoir and multiluser systems and considering a wide time horizon to describe hydrologic behaviors, system component requirements, and cost-benefit performance.

In recent decades significant progress has been made in the theoretical development and in the application of network flow models to real problems (Ahuja et al. 1993) so that network flow models have become alternative optimizers of linear programming to obtain fast solutions for minimum-cost water-allocation problems.

Besides, building an adequate model for system design and management optimization leads to problems that, though not of a pure network, can take advantage of the simple structure of most of the constraints in these problems. From previous investigations carried out to test the efficiency of different codes for solving network minimal cost flows (Sechi et al. 1994), it was seen that the development of specialized algorithms for network structured linear programming problems allows a fast solution even when using easily available, public-domain codes.

By using a network characterization of the water-resource system, in its basic formulation network linear programming finds a minimal cost solution for water flow commodities throughout the network. The network model has nodes that can represent demand, supply or simply transshipment centers; and arcs that represent a space transfer of water by means of works such as piping and canals, as well as a time transfer of water resources stored in reservoirs.

For a given configuration of the system where water-demands, resources and the dimensions of the works are fixed, network linear programming, as other general optimization techniques, obtains the best flow configuration performed by an ideal system-manager with a perfect knowledge of the time sequences of inflows and demands. Obviously no operating rules are retrieved even if a posteriori rules can be deduced from the final flow configuration.

Even if this network approach simply reaches an optimal flow configuration for a given system, with some additions it can also be used to face design problems. Network linear programming was used in this way by Kuczera (1989) to determine the minimum reservoir capacity that meets the demand over some planning period.

Considering that design variables interact with the network model by modifying problem constraints, Cao et al. (1988) faced the design optimization problem by a decomposition technique that leads to a main program, with no special constraints, that requires the solution of a simple linear network subprogram at each step.

Recently, a generalized network algorithm for water supply system optimization has been applied by Sun et al. (1995). Considering a matrix partition and network simplex method, the algorithm is able to exploit the network substructure in the model and handle nonnetwork type constraints and variables such as gains and losses of flows in the system.

A specialized version of the simplex technique, has also been proposed by Cao and Niedda (1990) for a problem of linear optimization with design variables. The implemented algorithm takes advantage of the simple structure of the constraints in order to reduce execution time and memory requirements to a reasonable amount even in the case of large problems.

In the case of nonlinear objective functions (OFs) and non-linear project constraints, an expansion procedure based on network programming has been proposed by Sechi and Zuddas (1995) using trade-offs between dimension of water works, reliability of the system, and prediction of severity in demand shortfalls.

As a matter of fact, linear assumptions are not always strictly valid in computations for the design and operation of real water-resource systems (Loucks et al. 1981). However, some approximations are usually introduced to reduce the model to linearity and reach at least preliminary solutions which may be improved in a subsequent simulation stage of the system performances.

For a linear design problem, a mixed optimization procedure based on network linear programming and the gradient method will be described hereafter. The problem formulation allows the use of network linear programming as a subproblem that assumes prefixed design variables and permits a gradient evaluation of their directions of improvement in order to reach a good approximation of the global OF optimum. It can also be seen that some kind of nonlinearities in the cost functions of design variables can also be handled with this approach so long as they guarantee the quasi-convexity of the global OF.

As will be shown in the following, the solution technique performs well in purely linear problems, since it requires much
shorter calculation times than in the specialized simplex method (Cao and Niedda 1990) to reach a near-optimal solution.

PROBLEM STATEMENT

The formal representation of a problem of optimal design and operation of a complex water resource system must take into consideration different variables that, as usual, can be classified into design variables, such as the volume of the reservoirs, the transfer-work capacity, and the extension of irrigation sites, and flow variables that represent the water transferred to meet the system requirements at each time step.

The problem formulation can be carried out conveniently with a network description of the system consisting of a finite set of nodes and a set of directed arcs. The construction of the single period network (basic graph) from the hydraulic scheme of the water system can be drawn as in the simple example shown in Fig. 1.

In the single period graph, arcs represent the transfer of water by means of the activity of each system component; nodes represent demand, supply or simply transshipment centers.

The multiperiod network in Fig. 2 can easily be obtained from the basic graph by replicating it by the number of periods in the considered time horizon. Note that each storage activity transfers water to the following time period by means of an arc connecting homologous nodes in adjacent periods.

The root node, denoted by U in Figs. 1 and 2 collects all flows actually consumed as well as spills. Moreover, arcs, associated with imported supply flows when a shortage occurs, are drawn from U to the demand nodes. Therefore, for the root node the related output is always equal to the algebraic sum of inputs and outputs for the nodes of the entire water system.

Let \( \mathcal{R} = (N, \delta) \) be the multiperiod network with node set \( N \) and arc set \( \delta \). Let us also define by \( y \) the vector in which components are the dimensions of design variables, and by \( x \) the vector in which components are the values of flow variables.

For simplicity, the problem of the optimal configuration of the water-resource system and the related optimal flows will be restricted to the case of ordinary cost-benefit analysis by finding the minimum value of an OF that should take into account costs given by the construction of works of dimension \( y \) (design variables) as well as those given by operating the system in \( T \) time steps and obtaining the \( x \) (flow variables) volumes of water transferred along arcs in the multiperiod network.

As usual, converting each cost term into comparable values, the general form of the o.f. for the cost-benefit linear problem can be written

\[
\min (cy + bx)
\]

Assuming fixed values for the design variables and facing the problem with network linear programming, the following two different types of constraint equations for flow variables can be derived (Cao and Niedda 1990): capacity constraints and demand constraints.

The former type of constraints is given by a subset \( y' \) of design variables referred to conveyance or storage works that produce an upper bound to the feasible flow values along the arcs that are associated to \( y' \) in the graph topology.

If we indicate with \( \delta' \subset \delta \) the arc set in the multiperiod network corresponding to the design variables \( y' \), and with \( u' \) the bound values for flows in a given set of arcs, the constraints can be synthesized

\[
(x \leq u' = G'y')_{k' \subset \delta'}
\]

Dealing with \( k = (1, K') \) design variables, \( G = \) matrix with \( K' \) columns and \((K' \times T) \) rows; each \( G_{ik} \) column vector gives the relation between the design variable \( y_i \in y' \) and the maximum flow allowed for each period in the corresponding arc.

Generally \( u' \) is a subvector of the flow upper bound vector \( u = (u', u'') \) where \( u'' \) can be derived from external limits to the arc-flows in the real problem.

The second type of flow constraint is given by the continuity or flow conservation conditions at that subset of nodes \( N' \) \( \subset N \) where required demands are linked to design variables. The output values at these nodes can be evaluated by a subset \( y'' \) of design variables referring to the dimension of water-demand centers as municipal zones, land irrigation, industry, etc. In the multiperiod network, as we change the values of design variables, the resulting node output evaluation modifies the flow continuity equations.

If we indicate with \( p' \) the flow demands determined by values of the design variables \( y'' \) at node set \( N' \subset N \) in the multiperiod network, the constraints can be written

\[
(E'x = p' = -(Hy''))_{k'' \subset \delta''}
\]

where the node-arc incidence matrix \( E' \) is extended to the \( N' \) node set. This matrix is made up of some of the rows of the incidence matrix \( E \) of the continuity constraints. As is known, in incidence matrices columns have only two non-null elements, that are \(-1 \) and \( 1 \), corresponding to the nodes \((i)\) and \((j)\) associated to the arc-column \((i, j)\).

The \( H \) matrix gives the relation between variable dimension and required output. \( H \) is made up of \( K'' \) columns and \((K'' \times T) \) rows; each \( H_{ik} \) column vector, \((k = 1, K'')\), gives the relations between the design variable \( y_i \in y'' \) and the node demands in each period.

The vector \( p' \) is a subset of the right side vector term \( p = (p', p'') \) in the node continuity equations. Hydrological inputs and fixed demands are set in \( p'' \); transshipment nodes without input or output are characterized with a null \( p \) value.

On the basis of the preceding remarks, the linear optimization problem for a given configuration \( y'' \) of design variables can be reduced to a network linear model

\[
\min z = bx; \quad \text{s.t.} \quad Ex = p; \quad x \leq u; \quad x \geq 0 \quad (4a-d)
\]

where, as seen before, subsets of the vectors \( p \) and \( u \) depend
on the current values of design variables \( y^* \), and \( E \) is the node-arc incidence matrix of the \( \Re \) multiperiod network. As previously remarked, this network linear model can be efficiently solved with specialized codes (Sechi et al., 1994). Therefore, the minimal-cost flows configuration of the water system can be easily evaluated solving (4) for different \( y = (y', y'') \) values.

Since, as usual, the flow-cost vector \( b \) can be considered constant for a wide range of design variable dimensions, and the subsets \( u'' \) and \( p'' \) are fixed on the basis of previous remarks, as a matter of fact the OF \([\text{4a}]\) is only a function of the design variables: \( z(y) \).

As a result, the global design-management optimization model can be written taking into account only the \( y \) variables

\[
\min f(y) = cy + z(y); \quad \text{s.t.} \quad y \leq s; \quad y \geq r \quad (5a-c)
\]

where the determination of the best \( y \) is only conditioned by upper and lower bounds constraints, while the evaluation of the OF requires the solution of the network linear problem (4).

**ANALYSIS OF OBJECTIVE FUNCTION**

It has already been stated that the first part of the objective function \([\text{5a}]\) refers to the costs of design works whereas the second part of the OF is related to the costs obtained from the management of the system. The latter has a less evident structure because of its formulation as a solution of an additional minimization problem.

Since the latter problem is expressed as a network linear problem, the optimum \( z(y^*) \) with respect to a pre-defined \( y^* \) can be examined considering the dual variables \( w \) of problem (4) (Bazaraa and Jarvis 1977)

\[
z(y^*) = wp + (b_n - wE_{ac})u_v
\]  
(6)

The elements with the \( N \) subscript are made up of the components of nonbasic arcs at their upper bound. The matrix \( E_{ac} \) that appears in the expression is made up of some of the columns of the incidence matrix \( E \) of the continuity constraints.

In the previous expression some terms depend on the value assumed by the vector \( y^* \). Grouping together under one constant the terms that do not depend on \( y^* \), the OF can be rewritten as follows:

\[
z(y^*) = wp' + (b_n - wE_{ac})u_v' + \text{Const.} \quad (7)
\]

In this expression the components of the vectors \( p' \) and \( u' \) depend on the vector \( y^* \) according to (2) and (3). Therefore, by substituting the following equation is obtained:

\[
z(y^*) = -wHy^* + (b_n - wE_{ac}G_{x,y})y' + \text{Const.} \quad (8)
\]

Since for a predefined \( y = y^* \) the problem of optimization can be solved by a network linear programming model, considering \( y \) as an independent variable, the function \( z(y) \) gives rise to a polyhedral convex surface made up of the intersection of hyperplanes associated with the basic solutions of the minimal cost network problem and valid for a set of \( y^* \) values with the same optimal basis. Beyond this set of values the optimal solution is made up of another basic configuration that gives rise to different values of the hyperplane coefficients.

Fig. 3(a) shows the surface \( z(y) \) in the case of the simple water resource system of Fig. (1) for which minimal cost flows are obtained considering as design variables the dimensions of a regulation reservoir and an irrigation area. The costs in the network problem only concern the penalties of irrigation shortfalls. A time horizon of 4 yr subdivided in two-month periods with 24 periods was used for the optimization.

Assuming a linear relation for the cost of design works, as in (5a) the surface \( f(y) = cy + z(y) \) remains polyhedral convex. In Fig. 3(b) the cost functions of the reservoir and the area have been assumed to be linear with respect to their size. The planes, each of which is an optimal basis, are clearly seen on this surface. They are few because of the small number of time steps considered.

On increasing the number of periods to be taken into account, both the number of the flow variables and that of optimal bases increase. As a consequence also the number of planes and their intersections increase. They become gradually closer until they resemble a curved surface. The same type of example extended to 54 yr with monthly steps (648 periods) generates the surface in Fig. 4.

The optimal surface \( f(y) \) remains convex if the design variable cost functions \( c(x) \) are expressed by non linear convex relations. If concave cost functions are taken into account, as \( c(y_i) \) sketched in Fig. 5(a), the proposed procedure could be applied if the strictly quasiconvex condition for \( f(y) \) is satisfied:
In order to guarantee the absence of local minima different from the global minimum (Bazaraa and Shetty 1979), as in the case reported in Fig. 5(b).

More serious problems arise if strongly concave cost functions are considered, as is often the case when the design variable size tends to zero, due also to fixed costs. These concave regions therefore must be outside the optimization domain of the design variables, which must be limited by the convex region of the cost function. This is not always a simple operation and a local investigation on the objective function will be necessary to assess the values assumed in the excluded regions of the domain.

**OPTIMIZATION PROCEDURE**

In the preceding section it was seen that the global optimization problem may only be expressed as a function of the design variables, that are far less numerous than the flow variables. For this reason, the problem may be adequately addressed by solution techniques with far better convergence characteristics than using the linear programming techniques applied to the global model.

As a matter of fact, the optimization problem (5a–c) formulated as a minimization of the cost function \( f(y) \) expressed only in terms of design variables \( y \), belongs to the category of nonlinear and nonconstrained multidimensional optimization problems (Bazaraa and Shetty 1977).

Moreover, for the previous reasons, the function \( f(y) \) is constant, in the space of \( y \), gives rise to a convex piecewise linear surface.

The solution methods for the problem can be divided into two categories according to whether they use local derivatives in determining the minimum search directions or not.

The methods that do not calculate derivatives, generally proceed by attempts at optimization along one coordinate at a time or along prefixed directions and using heuristic rules to determine the length of each optimization step.

The well-known steepest descent or gradient method, on the other hand, belongs to the group of methods that use the calculation of derivatives. The direction of steepest descent is determined by the gradient vector \( \nabla(\mathbf{f}(y)) \) and at each iteration the minimum is searched for along this direction.

As with the function \( f(y) \), the components of the gradient vector \( \nabla(\mathbf{f}(y)) \) are made up of the sum of the two addends

\[
\nabla(\mathbf{f}(y)) = \nabla(\mathbf{c}(y)) + \nabla(\mathbf{z}(y))
\]

The first term of the second member, in the considered linear case, is simply made up of the coefficients of the cost function with respect to each design variable. If a nonlinear cost function is considered, partial derivatives have to be calculated at the generic point \( y^* \).

The second term is a function of the design variables \( y' \) and \( y^* \), respectively through the subvectors of the upper bounds \( \mathbf{w}' \) and of the demands \( \mathbf{p}' \), relating to a predefined system configuration \( y^* \).

According to (8), the partial derivatives are determined by the equations

\[
\frac{\partial \mathbf{z}(y)}{\partial y_i} = (\mathbf{b}_i - w \mathbf{E}_w) g_{w,i}; \quad \frac{\partial \mathbf{z}(y)}{\partial y_i} = -w h_i
\]

where \( d_i \) stands for the set of nonbasic arcs \( (i,j) \), associated with the design variable \( y_i' \), at their upper bound.

As matrix \( \mathbf{E}_w \) in (11a) is made up of columns having only two non-null elements, that are \(-1\) and \(1\), corresponding to the nodes \((i)\) and \((j)\), (11a) can be rewritten as follows:

\[
\frac{\partial \mathbf{z}(y)}{\partial y_i} = \sum_{a \epsilon d_i} (b_a - w_i + w_j) g_{a}
\]

where the summation is extended to the set of nodes \( i \in N^s \), that, as mentioned in section (2), are associated to the design variable \( y_i' \).

On the basis of the analysis in the preceding section, however, for the examined optimization problem, the function \( f(y) \) is not differentiable everywhere.

The function \( \mathbf{z}(y) \), in the design variable space, admits one gradient vector \( \nabla \mathbf{z}(y) \) in the generic point \( y^* \), and is therefore differentiable in \( y^* \), only if the same optimal basis of the network problem remains optimal in the \( y^* \) neighborhood.

However, along the intersections between two or more planes that correspond to different optimal bases, the derivatives of the function \( \mathbf{z}(y) \) are no longer unique and the function at these points is no longer differentiable.

The notion of differentiability may be extended to this type of functions, as shown by Held et al. (1974), utilizing the definition of subgradient. To illustrate how the subgradient optimization method behaves, examples and computational experiences have also appeared in Sandi (1978).

A vector \( \mathbf{d} \) is defined as a subgradient in \( y^* \) of the convex function \( f \) if

\[
f(y) - f(y^*) \geq \mathbf{d}(y - y^*)
\]

where the points \( y^* \) are in the domain of \( f(y) \).

In the points in which the function is differentiable the direction of the search of the minimum coincides with the gradient, while at the points at which the function is not differentiable the direction could be chosen from among all the subgradients.

By extending the concept of differentiability with the subgradient approach, an optimization procedure that should take into account the value of the derivatives of the cost function \( f(y) \) can also be applied to these problems.

For the points of nondifferentiability, the strategy of not searching for the direction of maximum slope was chosen. Such a search would have entailed too much calculation. Instead, the value (10) that was readily available among the possible subgradients was used.

The reason for this is easily understood from an analysis of the surfaces of the objective functions reported in Figs. 3 and 4, shown previously. In fact, when the number of periods increases as a consequence of the growth of the number of optimal bases, the surface of the function tends to smoothen, because the number of related planes increases proportionately, thus rendering the search for the direction of maximum slope not only cumbersome but hardly useful.
This procedure is also different from the standard steepest descent method in another aspect, i.e., the determination of the step length in the predefined direction at each iteration. The length of the individual steps, as well known, can be established even without searching for any optimal value along the chosen direction. By seriously overloading the calculations this search is not particularly advantageous whenever the shape of the function to be minimized is not extremely regular (Held et al. 1974).

Generally speaking, starting from an initial feasible solution \( y_0 \), the sequence of solutions in the optimization process can be calculated by the following expression:

\[
y_{t+1} = y_t + t_j \nabla f(y_j), \quad j = 0, 1, \ldots \tag{15}
\]

where \( t_j \) = step length after \( j \) iterations. Theoretic conditions for the convergence to the minimum value of the sequence are the tendency of \( t_j \) to zero and of \( \sum t_j \) to infinity for \( j \) tending to infinity.

Following a widely used method, evaluating the step length we use the formula

\[
t_j = \lambda_j \frac{f^* - f(y_j)}{[\nabla f(y_j)]^2} \tag{16a}
\]

where

\[
0 < \lambda_j \leq \lambda_c; \quad f^* \leq f(y_j) \tag{16b,c}
\]

while the values of \( f^* \) and of \( \lambda_c \) can be updated iteratively during the optimization procedure.

For the case in which the value of \( f^* \) coincides with the minimum value of \( f \), it has been shown (Poljak 1967) that (15) converges to the optimal value, which can be reached with lower \( f^* \) values than the actual minimum of the function.

In conclusion, starting from an initial feasible solution of the design variables, the optimization procedure determines the sequence of solutions by means of (15). In this expression the first part of the gradient (10) is made up of the coefficients of the cost function of the design variables, while the second part is obtained from the solution of a network problem, by which the terms of (12) and (13) can be easily evaluated.

APPLICATION EXAMPLES

In order to give a first evaluation of the possibilities of application and of the performance of the optimization technique described in the previous sections, a simple example of water-resources system has been examined. The system is made up of two regulation reservoirs, a hydroelectric plant, a diversion dam and two irrigation areas. Its representation by a network generates the basic graph reported in Fig. 6 made up of 12 arcs and 6 nodes besides the root node \( U \).

As already mentioned, the multiperiod graph is constructed by reproducing the basic graph identically for each of the time steps in the reference period.

With a time span of 54 yr and a monthly step, a total of 648 periods is obtained. The multiperiod network is therefore made up of 3,888 nodes and 7,776 arcs.

The costs of the design variables have been assumed linear in order to compare the results with the optimal value determined by linear programming. The costs on the flows, that have been attributed to the network subproblem, have been assigned to the two deficit arcs that connect the root node with the two irrigation areas, while a benefit is attributed to the arc associated with the hydroelectric plant in order to take into account the benefits from the production of energy.

Initially the design variables only included a reservoir \( y_1 \) and an irrigation area \( y_5 \) while the remaining works were assumed fixed.

The \( f(y_1, y_5) \) surface that is to be minimized is represented in Fig. 7, where the isocost curves are reported.

The optimization procedure has been applied starting from a generic point on the frontier \( (y_1 = 400, y_5 = 1,000) \), assuming a value of \( f^* = -60,000 \) lower than absolute minimum and utilizing an initial value of \( \lambda = 2 \).

The solution path reported in Fig. 7 has been obtained by applying the empirical rule of dividing \( \lambda \) by 1.5 every 3 iterations.

After 50 iterations the objective function \( f(y) \) in the final situation was \(-49,393\), which was near the optimal value, \(-49,396\), obtained by linear programming.

During the procedure, when the step length was such that the solution was outside the admissible region (field of definition of the figure), we used the technique of restarting from the nearest frontier point onto which the obtained solution was projected.

Subsequently, all six design variables in the example were considered, and the optimal dimension for each was searched using the same values of \( f^* \) and \( \lambda_c \) given before.

Using the technique illustrated above, after 50 iterations the objective function was determined as being \(-45,423\), as compared to the absolute minimum of \(-45,468\) obtained by linear programming. In Fig. 8 the dots stand for the successive solutions obtained with the proposed mixed optimization technique, while the continuous line indicates the results of a specialized simplex optimization method (Cao and Niedda 1990) for the solution of the same problem.

The abscissa may be considered the time axis on which the performances of the two applied methodologies are presented. With an HP9000/730 computer, the specialized simplex method took 321 s while with the mixed optimization technique it was reduced to 98 s.

Finally, in the preceding example, nonlinear cost laws were

FIG. 6. Examined Water-Resource System

FIG. 7. Solution Path in Optimization Procedure
introduced for all six design variables reported in the three graphs in Fig. 9, respectively, for the two regulation reservoirs, for the hydroelectric plant and the diversion dam, and for the two irrigation areas.

In the figure the admissible regions of variability for each design variable are also indicated; on the basis of preliminary evaluations and in order to guarantee the convergence of the optimization procedure, only convex regions for the cost functions were considered. A local analysis of the OF is necessary at least near the zero value for reservoir dimensions in order to evaluate local minima due to concave parts of the cost functions and to fixed costs. Note that also in the case of linear approximation with fixed initial costs for these cost functions, a local analysis will be drawn for the zero value. Aware of these precautions, the application of the proposed mixed optimization procedure did not involve any substantial changes with respect to the linear case.

To test the validity of the obtained results, linear programming methodologies were obviously not applicable and the objective function was tested with a grid method to search for its minimum value.

By restricting the grid size in the optimum area for each variable, after many calculation hours, we obtained the minimum value of $-45,984$ for the objective function compared to $-45,968$ obtained with only 50 iterations in the proposed optimization procedure.

**CONCLUDING REMARKS**

The proposed mixed optimization technique takes advantage of the peculiarities of the formal structure of the model for the problem of optimal design and management of water resource systems.

The model formulation allows to use network linear programming as a subproblem that assumes the knowledge of design variables and permits a gradient evaluation of the directions of improvement for design variables until a near-optimal configuration is reached. As the o.f. of the network problem is not differentiable everywhere, a subgradient extension in the optimization procedure is required.

Taking into account the preceding, as expected, in the examined applications the optimization technique performs well in the pure linear case. Near-optimal results are reached in a few number of iterations and require much shorter calculation times than the other optimization methods considered.

Moreover, the technique seems flexible in addressing different design and management problems by allowing to introduce nonlinear convex cost relations for the design works. If concave parts of cost functions are neglected, there is no guarantee of optimality of obtained results. Further investigations are needed in these cases to find possible local minima different from the one obtained.

Because only quasiconvexity in the global o.f. is required...
applying the procedure, more investigations are needed to verify the level of concave nonlinearities that can be taken into account in the procedure.

APPENDIX. REFERENCES


